Standard Cantor set. Minkowski dimension.

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hicago lectures

Cantor set.

1) Definitions. $C_0 = I$.

a) C_n – obtained from C_{n-1} by dividing each interval in 3, and removing the middle part

(b) $T: C_1 \to C_0$ $T(x) = \begin{cases} 3x, & 0 \le x \le \frac{1}{3} \\ 3x-2, & \frac{2}{3} \le x \le 1 \end{cases}$ Discuss later.

(b') $T_1(x) = \frac{1}{3}x$, $T_2(x) = \frac{1}{3}x + \frac{2}{3}$, $C = T_1(C) \cup T_2(C)$

$C = \{x : T^n x \in I \ \forall n\}$ – attractor of this system.

(c) Trinary decomposition

$x \in C \iff \exists \ x = 0.x_1x_2 \ldots x_n \ldots, \ \forall n \ x_n \ne 1$

2) How large is C? Measure (Lebesgue) is zero.

But it just means that 1D Lebesgue measure is wrong here. If you try to measure interval by 2D Lebesgue measure, you get 0. So, what is the right measure? Dimension?

3) First attempt: Minkowski dimension

K – closed subset of a metric space (X, d) which is totally bounded, i.e. $\forall \varepsilon > 0$

$N(\varepsilon, K) := \min\{n : \exists n \text{ sets } X_1, \ldots, X_n, \ \operatorname{diam} X_j \le \varepsilon, \ K \subset \bigcup X_j\} < \infty$.

Minkowski dimension – rate of growth:

$N(\varepsilon, K) \sim \frac{1}{\varepsilon}^{-\operatorname{Mdim} K}$, or, more rigorously

$\operatorname{Mdim} K = \lim \frac{\log N(\varepsilon, K)}{\log \frac{1}{\varepsilon}}$ – ~~[illegible]~~ (equivalently:

[illegible]

sometimes, upper and lower $P(\varepsilon, K) = \max(n : x_1, \ldots, x_n \in K, \ \operatorname{dist}(x_i, x_j) \ge \varepsilon)$

$P(2\varepsilon, K) \le N(\varepsilon, K) \le P(\frac{\varepsilon}{2}, K)$

so

$$\overline{\lim} \frac{\log P(\varepsilon, K)}{\log \frac{1}{\varepsilon}} = \overline{\lim} \frac{\log N(\varepsilon, K)}{\log \frac{1}{\varepsilon}}$$

The same with $\underline{\lim}$.

P is used for lower estimates, N – for upper.

1) I^d – dimension d.

2) C $\quad P(\varepsilon, K) \ge 2^{n}$ if $3^{-n} \ge \varepsilon \ge 3^{-n-1}$

$N(\varepsilon, K) \le 2^n$ if $3^{-n+1} \ge \varepsilon \ge 3^{-n}$ so $\lim \frac{\log N(\varepsilon, K)}{\log \frac{1}{\varepsilon}} = \lim \frac{P(\varepsilon, K)}{\log \frac{1}{\varepsilon}} =$

$\lim_{\infty} \frac{n \log 2}{n \log 3} = \frac{\log 2}{\log 3}$.

3) $K = \{0\} \cup \bigcup_{n=1}^{\infty} \{\frac{1}{n}\}$ – closed set.

$P(\varepsilon, K) \ge \frac{1}{\sqrt{\varepsilon}}$. since for $(N+1)^{-2} \le \varepsilon \le N^{-2}$, then for $n < N$, $\frac{1}{n} - \frac{1}{n-1} > \frac{1}{n(n-1)} \ge \varepsilon$

But so we need $[0, \frac{1}{N+1}]$ can be covered by $N+1$ intervals of length ε.

so $N(K, \varepsilon) \le 2\varepsilon^{-\frac{1}{2}} + 1$.

Thus $\operatorname{Mdim} K = \frac{1}{2}$.

Discuss a more general example. Cantor set in R^d. n-th step: 2^{nd} cubes of size l_n.

Same as for the usual Cantor set

$\overline{\operatorname{Mdim}} = \overline{\lim} \frac{nd \log 2}{|\log l_n|}$,

$\underline{\operatorname{Mdim}} = \underline{\lim} \frac{nd \log 2}{|\log l_n|}$,

since for $l_n \le \varepsilon \le l_{n-1}$,

$N(\varepsilon, K) \le 2^n$

$P(\varepsilon, K) \ge 2^{n-1}$

Hausdorff measure and dimension.

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What was wrong with Mink. dimension? Try to make a measure from it.
Cover K by sets of the same diameter ε. (Or at most ε).
$M_\alpha(K) = \inf \sum_{j=1}^{n} \varepsilon^\alpha = \inf N(\varepsilon, K)\varepsilon^\alpha$ – Minkowski content. Then $M_\alpha(K) = \begin{cases} 0, & \alpha > \overline{\text{Mdim}}\, K \\ >0, & \alpha < \text{Mdim}\, K \end{cases}$ and we can describe $\underline{\text{Mdim}}\, K$ as
$\underline{\text{Mdim}}\, K = \sup\{\alpha : \overline{M_\alpha(K)} = 0\} = \inf\{\alpha : M_\alpha(K) > 0\}$

But, as we know, taking the sets of the same diameter does not work! For Lebesgue measure in d dimension we took
$m_d(K) = \inf\{\sum \varepsilon_j^d : K \subset \cup K_j,\ \text{diam}\, K_j = \varepsilon_j\}$. So, let us do the same for an arbitrary α. α-dimensional Hausdorff content is defined as

$$H_\alpha(K) := \inf\{\sum \varepsilon_j^\alpha : K \subset \cup K_j,\ \text{diam}\, K_j = \varepsilon_j\}$$

Can even generalize it slightly. Let $h(t) \geq 0$ be a strictly increasing continuous function on $\mathbb{R}_+$, $h(0) = 0$. Gauge function. Define h Hausdorff content as
$$H_h(K) := \inf\{\sum h(\varepsilon_j) : K \subset \cup K_j,\ \text{diam}\, K_j = \varepsilon_j\}.$$
Same as what we had before for $h(t) = t^\alpha$.

Note that if K is countable, $H_h(K) = 0$ (because for any ε we can cover $a_j \in K$ by disk of diameter ε_j with $h(\varepsilon_j) < 2^{-j}\varepsilon$.

Lemma 1. If $H_h(K) = 0$ and $\lim\limits_{t \to 0} \dfrac{g(t)}{h(t)} < \infty$, then $H_g(K) = 0$.

Proof $\forall \varepsilon > 0\ \exists$ covering K_j of K such that $\sum h(\text{diam}\, K_j) < \varepsilon$.
Then, since h is strictly increasing, $\max \text{diam}\, K_j \to 0$ as $\varepsilon \to 0$. Thus for some C, $g(\text{diam}\, K_j) < C h(\text{diam}\, K_j)$, so $\sum g(\text{diam}\, K_j) < C\varepsilon \Rightarrow H_g(K) < C\varepsilon$ ■

Corollary 2. If $H_\alpha(K) = 0$ and $\beta > \alpha$ then $H_\beta(K) = 0$.
If $H_\alpha(K) > 0$ and $\beta < \alpha$, then $H_\beta(K) > 0$.

Similarly to the discussion of (lower) Minkowski dimension, can now define Hausdorff dimension as
$$\text{Hdim}\, K := \inf\{\alpha : H_\alpha(K) = 0\} = \sup\{\alpha : H_\alpha(K) > 0\}$$

Of course, $\text{Hdim}\, K \leq \underline{\text{Mdim}}\, K \leq \overline{\text{Mdim}}\, K$ (For any $\alpha > \underline{\text{Mdim}}\, K$, cover by $N(\varepsilon_j, K)$ balls of radius ε_j, get $N(\varepsilon_j, K)\varepsilon_j^\alpha \to 0$).

One problem with H_h – it is not a measure.
Example. $H_{1/2}([0,1)) = H_{1/2}([1,2)) = 1$, but $H_{1/2}([0,2)) = \sqrt{2} < H_{1/2}([0,1)) + H_{1/2}([1,2))$. (simply because $a^{1/2} + b^{1/2} \geq (a+b)^{1/2}$)

To make it into a measure, force covering by smaller and smaller sets:

$$m_h^\varepsilon(K) := \inf\{\sum h(\varepsilon_j) : K \subset \cup K_j,\ \text{diam}\, K_j = \varepsilon_j < \varepsilon\}$$
and
$m_h(K) := \lim\limits_{\varepsilon \to 0} m_h^\varepsilon(K)$. The limit always exists (as a limit of an increasing function), but can be infinite.

m_h satisfies the following properties.

1) Monotonicity:
$K_1 \subset K_2 \Rightarrow m_h(K_1) \leq m_h(K_2)$ – obvious, any cover of K_1 is a cover of K_2.

2) Subadditivity (countable)
$m_h\left(\bigcup_{j=1}^{\infty} K_j\right) \leq \sum_{j=1}^{\infty} m_h(K_j)$ Pf Select cover of K_j with $\sum h(\varepsilon_j) < m_h^\varepsilon(K_j) + \varepsilon 2^{-j}$, $\varepsilon_j < \varepsilon$ union of the covers will make the cover of $\cup K_j$ with $\sum h(\varepsilon_j) < \sum m_h^\varepsilon(K_j) + \varepsilon$. Let $\varepsilon \to 0$. ■

3) Metric separated additivity:
if $\text{dist}(K_1, K_2) > 0$ then $m_h(K_1) + m_h(K_2) = m_h(K_1 \cup K_2)$.
Pf. When $\varepsilon < \text{dist}(K_1, K_2)$, the covers of K_1 and K_2 do not know about each other ■

Def. A set function satisfying 1)–3) is called Metric Outer measure.

Thm. Any metric outer measure restricted to Borel sets is a measure.
No proof!

Property $m_h(K) \geq H_h(K)$ and $H_h(K) = 0 \Leftrightarrow m_h(K) = 0$

Proof The first statement follows from the definition.

For the second notice that if K_i is a covering such that $\sum h(\operatorname{diam} K_i) < \varepsilon$ then $\max \operatorname{diam} K_i < h^{-1}(\varepsilon) \to 0$ as $\varepsilon \to 0$.
so $m_{h, h^{-1}(\varepsilon)}(K) < \varepsilon$. Let $\varepsilon \to 0$. ∎

Lemma. If h, g are two gauge functions, $\lim_{t \to 0} \frac{h(t)}{g(t)} = 0$, and $m_g(K) < \infty$ then $m_h(K) = 0$

Pf. Fix any $\delta > 0$ and choose $\varepsilon > 0$ such that $t < \varepsilon \Rightarrow h(t) < \delta g(t)$.
Consider a covering of K by K_i such that $\operatorname{diam} K_i < \varepsilon$ and $\sum g(K_i) < m_{g,\varepsilon}(K) + \varepsilon$. Then
$m_{h,\varepsilon}(K) \leq \sum h(K_i) < \delta \sum g(K_i) < \delta (m_{g,\varepsilon}(K) + \varepsilon)$.
Now let $\varepsilon \to 0$ to see that $m_h(K) < \delta$. ∎

Corollaries:

1) If $\lim_{t \to 0} \frac{h(t)}{g(t)} = 0$, $m_h(K) > 0$, then $m_g(K) = \infty$

2) If $\alpha = \operatorname{Hdim} K$ and $\beta < \alpha < \gamma$, then $m_\beta(K) = \infty$ and $m_\gamma(K) = 0$

Thus $\operatorname{Hdim}(K) = \inf\{\alpha : m_\alpha(K) = 0\} = \sup\{\alpha : m_\alpha(K) = \infty\}$.

So we can estimate the Hausdorff dimension from above, by presenting a cover. How to estimate it bellow?

Def. A measure μ is called h-smooth if for some C and for every ball $B(x,r)$, $\mu(B(x,r)) \leq C h(r)$.

Thm (Mass distribution principle). Let $\mu(K) > 0$ for some h-smooth measure, then $m_h(K) \geq H_h(K) > \frac{\mu(K)}{C}$, where C is the constant in the definition of h-smoothness

Proof. Let (K_i) be any cover of K. Then $K_i \subset B(x_i, \operatorname{diam} K_i)$ for any $x_i \in K_i$.
Then $\mu(K) \leq \sum \mu(K_i) \leq \sum C h(\operatorname{diam} K_i)$. Take inf over all the coverings. ∎

Corollary. If $\mu(K) > 0$ for some α-smooth measure $(\mu(B(x,r)) \leq C r^\alpha)$ then $\operatorname{Hdim} K \geq \alpha$.

Using this, it is easy to prove that $\operatorname{Hdim} C = \frac{\log 2}{\log 3}$ (C - the usual Cantor set).
Construct μ by assigning $\mu(I_n^k) = 2^{-n}$ for any interval $I_n^k \in C_n$. $\mu(C) = 1$, and notice that for $3^{-n} \leq r < 3^{-n+1}$, $B(x,r)$ intersects at most one I_{n-1}^k, so $\mu(B(x,r)) \leq 2^{-n+1} \leq 2 r^{\frac{\log 2}{\log 3}}$, so μ is $\frac{\log 2}{\log 3}$-smooth.
Thus $\frac{\log 2}{\log 3} \leq \operatorname{Hdim} C \leq \operatorname{Mdim} C = \frac{\log 2}{\log 3}$.

Plan of the second lecture

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Last time:

2 notions of dimension: Mdim (upper and lower versions)
Hdim

$\mathrm{Hdim} \le \underline{\mathrm{Mdim}} \le \overline{\mathrm{Mdim}}$

Hausdorff measure m_h, H_h - content, $m_h \geq H_h$, $H_h = 0 \Rightarrow m_h = 0$

Tool for the lower estimate: mass distribution principle.

General form: $\exists\, \mu : \mu(K) > 0$, $\mu(B(x,r)) \le h(r) \Rightarrow H_h > 0$

Hdim form: $\exists\, \mu : \mu(K) > 0$, $\mu(B(x,r)) \le r^d \Rightarrow \mathrm{Hdim}\, K \geq d$.

Today

1) Frostman Lemma (inverse to MDP):
$K \subset \mathbb{R}^d$, $H_h(K) > 0 \Rightarrow \exists\, \mu : \mu(K) \geq H_h(K)$, $\mu(B(x,r)) < C_d\, h(r)$.

2) Furstenberg Lemma: a $\overset{\text{sufficient}}{\text{criterium}}$ on when
$\mathrm{Hdim} = \underline{\mathrm{Mdim}} = \overline{\mathrm{Mdim}}$

Let $T_b : [0,1] \to [0,1] : x \to bx \pmod 1$
same as b-adic shift. $x = \sum_{j=1}^{\infty} x_j b^{-j} \to T_b(x) = \sum_{j=1}^{\infty} x_{j+1} b^{-j}$.

If $K \subset [0,1]$, compact, $T_b(K) = K \Rightarrow \mathrm{Mdim}\, K = \mathrm{Hdim}\, K$.

Example. C - st. Cantor set, $T_3 C = C$.

3) Iterated function systems.

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Dimension for self-similar attractors. Open set condition.

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Def A map $T:(X,\rho)\to(X',\rho')$ is called uniform contraction if $\exists\, r<1$: $\rho'(Tx,Ty)=r\rho(x,y)$.

Example: rotation with scaling.

Theorem: Let $(T_1,\dots,T_k)$ be a family of uniform contractions, X – complete. Then $\exists!\, K\neq\emptyset$, compact, such that $K=\bigcup_{i=1}^{k}T_i(K)$. Moreover, for any $\vec p=(p_1,\dots,p_k)$ – probability vector, $\exists\, \mu_{\vec p}$: $\mu_{\vec p}=\sum_{i=1}^{k}p_i(T_i)_*\mu_{\vec p}$ $(\mu_{\vec p}T_i^{-1})$

(True even for non-uniform contractions, i.e. just $\leq r\rho(x,y)$). K is called an attractor of the system $(T_1,\dots,T_k)$.

Examples: 1) Cantor set: 2 with ratios $\frac{1}{3}$.

2) Von Koch Snowflake

4 with ratio $\frac{1}{3}$

Let K be an attractor of $(T_1,\dots,T_k)$ with contraction ratios $(r_1,\dots,r_k)$ correspondingly. The self-similarity dimension of K is defined as the unique α such that $r_1^\alpha+\dots+r_k^\alpha=1$. ($\alpha$ exists and unique since $f(\alpha):=r_1^\alpha+\dots+r_k^\alpha$ is strictly increasing, $f(0)=k\geq 1$, $f(\alpha)\to 0$ as $\alpha\to\infty$). For our examples: Cantor set $\alpha=\frac{\log 2}{\log 3}$. Von Koch $\alpha=\frac{\log 4}{\log 3}$.

Lemma: For such a K:

1) $H_\alpha(K)=m_\alpha(K)<\infty$

2) For any m_α-measurable subset $E\subset K$, $m_\alpha(E)=H_\alpha(E)$.

3) $H_\alpha(T_i(K)\cap T_j(K))=0$, $i\neq j$.

Proof 1) We only need to prove $H_\alpha(K)\geq m_\alpha(K)$ – opposite always true. Take any cover such that $\sum(\operatorname{diam}E_j)^\alpha\leq H_\alpha(K)+\varepsilon$. Now pick n large enough so that $(\max\operatorname{diam}E_j)(\max r_i)^n<\varepsilon$. Then $T_\sigma E_j$, where σ is a multi-index of size n, $\sigma=(\sigma_1,\dots,\sigma_n)$, and $T_\sigma:=T_{\sigma_1}\circ T_{\sigma_2}\circ\dots\circ T_{\sigma_n}$ is a cover of K with $\max\operatorname{diam}T_\sigma E_j<\varepsilon$. So

$$\text{Also}\ \sum_\sigma\sum_j(\operatorname{diam}T_\sigma E_j)^\alpha=\left(\sum r_i^\alpha\right)^n\sum(\operatorname{diam}E_j)^\alpha=\sum(\operatorname{diam}E_j)^\alpha.$$

$m_{\alpha,\varepsilon}(K)\leq H_\alpha(K)+\varepsilon$. Let $\varepsilon\to 0$.

2) Note that $H_\alpha(K)\leq H_\alpha(E)+H_\alpha(K\setminus E)\leq m_\alpha(E)+m_\alpha(K\setminus E)=m_\alpha(K)=H_\alpha(K)$ ∎

3)
$$m_\alpha(K)=m_\alpha\left(\bigcup T_i(K)\right)\leq\sum m_\alpha(T_i(K))=\sum r_i^\alpha m_\alpha(K)=m_\alpha(K).$$
so $m_\alpha(T_i(K)\cap T_j(K))=0\quad\forall\, i\neq j$ ∎

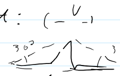
Tempting to say: $\alpha=\operatorname{Hdim}K$. Not always:

$T_1=\frac{2}{3}x$, $T_2=\frac{2}{3}x+1$. Self-similarity dimension is $2(\frac{2}{3})^\alpha=1\Rightarrow$ $\alpha=\frac{\log\frac{1}{2}}{\log\frac{2}{3}}>1$ – cannot be subset of $\mathbb{R}$!

Def. A family of maps $(T_1,\dots,T_k)$ satisfy Open Set Condition (OSC) if $\exists$ bounded non-empty open V such that $T_i(V)\subset V\ \forall i$ and $\forall i\neq j$ $T_i(V)\cap T_j(V)=\emptyset$.

Examples: Cantor set: $(-\overset{V}{\ }-)$

Von Koch



V – open triangle.

Does not have to contain K!

Theorem. Let K be an attractor of OSC family $(T_1,\dots,T_k)$ of uniform contractions of $\mathbb{R}^d$, α – its self-similarity dimension. Then $0<m_\alpha(K)<\infty$ and $\operatorname{Hdim}K=\operatorname{Mdim}K=\alpha$.

Pf First, let us define a measure μ for a probability vector $(r_1^\alpha,\dots,r_k^\alpha)$, i.e. $\mu(T_\sigma K)=r_{\sigma_1}^\alpha\cdots r_{\sigma_n}^\alpha$.

Prelim on μ and Σ_ε

We will show that μ is α-smooth, and, by MDP, $m_\alpha(K)>0$ (We already know that $m_\alpha(K)<\infty$)

To this end, fix $\varepsilon>0$ and consider the set Σ_ε of all multi-indices σ which satisfy $r_{\sigma_1}\cdots r_{\sigma_n}<\varepsilon\leq r_{\sigma_1}\cdots r_{\sigma_{n-1}}$. Σ_ε is an example of a Stopped set: for each infinite sequence $(\sigma_1,\dots,\sigma_n,\dots)$ we keep adding elements till certain condition is satisfied. Observe, that for each stopped set Σ, $(T_\sigma K)_{\sigma\in\Sigma}$ form a covering of K. Also, by induction the size of Σ, and the fact that $\sum r_i^\alpha=1$ and $\sum r_j^\alpha\mu T_j^{-1}=\mu$, one easily see that $\sum_{\sigma\in\Sigma}r_{\sigma_1}^\alpha\cdots r_{\sigma_n}^\alpha=1$ and $\mu=\sum_{\sigma\in\Sigma}r_{\sigma_1}^\alpha\cdots r_{\sigma_n}^\alpha\mu T_\sigma^{-1}$.

Returning to Σ_ε, observe that

$$\min r_j\cdot\varepsilon\operatorname{diam}K<\operatorname{diam}T_\sigma K\leq r_{\sigma_1}\cdots r_{\sigma_n}\operatorname{diam}K\leq\varepsilon\operatorname{diam}K$$

Using OSC to establish lower bound.

Now observe that if V – open set from OSC, then $\overline{V}\supset\bigcup T_i(\overline{V})$, so $K\subset\overline{V}$ (T_i – uniform contractions of complete $\overline{V}$)

Let V ~~contain~~ some ball of radius a.

Then $\{T_\sigma V,\ \sigma\in\Sigma\}$ are disjoint and

Then $\{T_\sigma U, \sigma\in\Sigma_\varepsilon\}$ are disjoint, each contain a ball of radius $a\,\varepsilon(\min r_i)$

Let us pick any ball $B(x,\varepsilon)$, $x\in K$. Then

if for some x, $\{\sigma\in\Sigma_\varepsilon, T_\sigma U\cap B(x,\varepsilon)\neq\emptyset\} = \{\sigma\in\Sigma_\varepsilon : T_\sigma U\cap B(x,\varepsilon)\neq\emptyset\}$

$T_\sigma U\subset B(x, \varepsilon(1+\operatorname{diam}(U)))$, and $\operatorname{Vol}(T_\sigma U)\geq c(d,a,\min r_i)\cdot\varepsilon^d$.

But $T_{\sigma_1}U\cap T_{\sigma_2}U=\emptyset$ if $\sigma_1,\sigma_2\in\Sigma_\varepsilon$. Thus

$\#\{\sigma : T_\sigma U\cap B(x,\varepsilon)\neq\emptyset\}\cdot c(d,a,\min r_i)\,\varepsilon^d\leq \operatorname{Vol}(B(x,\varepsilon(1+\operatorname{diam}U)))\leq c'\varepsilon^d$.

so $\#\{\sigma:\dots\}\leq C_2$.

so $\mu(B(x,\varepsilon)) = \sum_{\sigma\in\Sigma_\varepsilon}(r_{\sigma_1}\dots r_{\sigma_n})^\alpha\mu(\sigma_1^{-1}\circ\dots\circ\sigma_n^{-1}(B(x,\varepsilon)))\leq\varepsilon^\alpha\#\{\sigma:\dots\}\leq C\varepsilon^\alpha$.

so, by Mass Distribution Principle, $m_\alpha(K)>0$.

Using Σ_α to establish upper bound.

Now let us prove that $\overline{\operatorname{Mdim}}\leq\alpha$.

The sets

$\{T_\sigma K, \sigma\in\Sigma_\varepsilon\}$ form a cover of K, by sets of diam $\leq\varepsilon\operatorname{diam}K$

so $N(\varepsilon\operatorname{diam}K; K)\leq\#\Sigma_\varepsilon$. But

$1=\sum_{\sigma\in\Sigma_\varepsilon}(r_{\sigma_1}\dots r_{\sigma_n})^\alpha\geq(\varepsilon\min r_i)^\alpha\#\Sigma_\varepsilon$, so

$\#\Sigma_\varepsilon\leq(\varepsilon\min r_i)^{-\alpha}$, thus

$\overline{\operatorname{Mdim}}\,K=\overline{\lim}\dfrac{\log N(K,\varepsilon\operatorname{diam}K)}{\log(1/\varepsilon\operatorname{diam}K)}\leq\alpha$

It follows from 1) that $\alpha=\operatorname{Hdim}K\leq\underline{\operatorname{Mdim}}\,K\leq\overline{\operatorname{Mdim}}\,K\leq\alpha$

Remark. OSC is necessary: $m_\alpha(K)>0\Rightarrow$ OSC! (Schief, 1994)

Some sets defined by digits. Furstenberg Lemma

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An alternative approach is given by

Furstenberg lemma. Let $b \in \mathbb{N}$, $b > 2$, and

$T_b(x): [0,1] \to [0,1]$, $x \mapsto bx \pmod 1$ – b-adic shift.

Assume $K \subset [0,1]$ – T_b-invariant compact ($T_b(K) = K$). Then $\mathrm{Hdim}\,K = \mathrm{Mdim}\,K$

Proof. We always have $\mathrm{Hdim}\,K \le \overline{\mathrm{Mdim}}\,K$, so let us prove the opposite

Encode every b-adic interval by b-multiindex (and every $x \in [0,1]$ by sequence $(x_1, \dots, x_n, \dots)$, $x_j \in \{0, \dots, b-1\}$ via $x = \sum x_j b^{-j}$).
Fix $\alpha > \mathrm{Hdim}\,K$. $\exists$ covering (k_i) of K by intervals with $\sum (\mathrm{diam}\,k_i)^\alpha < \frac{1}{2(b+1)}$. Each k_i can be covered by at most $(b+1)$ b-adic intervals of $\mathrm{diam} \le \mathrm{diam}\,k_i$,
so there is a covering of K by b-adic intervals I_{σ_i} with $\underbrace{\sum (\mathrm{diam}\, I_{\sigma_i})^\alpha}_{\sum b^{-\alpha\, \mathrm{length}(\sigma_i)}} < (b+1)^\alpha \cdot \frac{1}{2(b+1)} < \frac{1}{2}$.
Let $\Sigma = \{\sigma_i\}$ – collection of the multi-indexes from this covering, $M = \max_{\sigma \in \Sigma} (\mathrm{length}\,\sigma)$.

Assume now that $\mathrm{length}\,\sigma > M$ and $I_\sigma \cap K \ne \emptyset$. Then, since $T_b K = K$, we can apply shift to σ till we get an element of Σ! (since they cover K). Thus $\sigma = \tau_1 \sigma_1$, where $\sigma_1 \in \Sigma$.
If $\mathrm{length}\,\tau_1 > M$, we can continue to get $\tau_1 = \tau_2 \sigma_2$, so $\sigma = \tau_2 \sigma_2 \sigma_1$.
Continuing with this logic, we get $\sigma = \tau \sigma_k \dots \sigma_1$ for any σ with $\sigma_i \in \Sigma$, and $\mathrm{length}\,\tau \le M$ (thus, there are only b^M choices of τ).

Now let us observe that

$$(*) = \sum_{\sigma : I_\sigma \cap K \ne \emptyset} b^{-\alpha\, \mathrm{length}\,\sigma} = \sum_{\tau,\, \mathrm{length}\,\tau \le M} b^{-\alpha\, \mathrm{length}\,\tau} \sum_{n=0}^{\infty} \sum_{\sigma_1, \dots, \sigma_n \in \Sigma} b^{-\alpha\, \mathrm{length}(\sigma_1 \dots \sigma_n)} \le b^M \sum_{n=0}^{\infty} \left(\sum b^{-\alpha\, \mathrm{length}\,\sigma_i} \right)^n \le \frac{b^M}{1 - \frac{1}{2}}$$

$$= 2b^M < \infty$$

By note that $\#\{\sigma : I_\sigma \cap K \ne \emptyset,\ \mathrm{length}\,\sigma = n\} \ge N(K, b^{-n})$.

Thus $(*) \ge \sum b^{-\alpha n} N(K, b^{-n})$, which means that $b^{-\alpha n} N(K, b^{-n}) \to 0$, so $\underbrace{\overline{\lim} \frac{\log N(K,\varepsilon)}{|\log \varepsilon|}}_{\overline{\mathrm{Mdim}}\,K} \le \overline{\lim} \frac{\log N(K, b^{-n})}{\log b^{-n+1}} \le \alpha$ ■

As an immediate application, we see that, since $T_3 C = C$, $\mathrm{Hdim}\,C = \mathrm{Mdim}\,C = \frac{\log 2}{\log 3}$

Let us now consider certain sets defined by digits.

Let A be a $b \times b$ matrix with $\{0,1\}$ entries.

$x \in C_A$ if $x = \sum x_j b^{-j}$ (b-adic form) and $\forall j$ $A_{x_j x_{j+1}} = 1$.

so $C = C_A$ for $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

We also assume that if j-th row of matrix is identically 0, so is j-th column – to allow continuation of any sequence not started with j;
C_A is T_b-invariant, so, by Furstenberg, $\mathrm{Hdim}\,C_A = \mathrm{Mdim}\,C_A$.
Let us compute $\mathrm{Mdim}\,C_A$.
We can simply count $N_n = \#\{\sigma : I_\sigma \cap C_A \ne \emptyset,\ \mathrm{length}\,\sigma = n\}$, which is
$\ge N(C_A, b^{-n})$ and $\le P(C_A, \frac{1}{2} b^{-n})$. This is the same as $\#\{(x_1 \dots x_n) : A_{x_i x_{i+1}} = 1\ \forall i < n\}$.

Note that $A^n_{x_1 x_n} = \#\{\sigma : \sigma_1 = x_1, \sigma_n = x_n, A_{\sigma_i \sigma_{i+1}} = 1\}$.

so $N_n = \sum_{i=1}^{b} \sum_{j=1}^{b} A^n_{ij}$. And $\mathrm{Mdim}\,C_A = \lim_{n \to \infty} \frac{\log N_n}{n \log b} = \frac{\log \lim_{n\to\infty} N_n^{\frac{1}{n}}}{\log b} = \frac{\log \rho(A)}{\log b}$, where

$\rho(A) = \lim_{n \to \infty} \left(\sum_{i=1}^{b} \sum_{j=1}^{b} A^n_{ij} \right)^{1/n}$ – spectral radius of A.

Frostman Lemma.

June 8, 2017 8:05 PM

Let us now formulate an inverse to Mass Distribution principle.

Thm (Frostman Lemma). Let h be a gauge function, $H_h(K) > 0$ for some $K \subset \mathbb{R}^d$. Then $\exists \mu : \mu(K) \geq H_h(K)$, $\mu(B(x,r)) < C_d h(r)$, for some C_d depending only on d.

Proof. $\forall$ binary cubes. Note that if we consider any covering of K by the binary cubes of sizes 2^{-n_i}, then $\sum h(2^{-n_i}) \geq \frac{H_h(K)}{\ell_d}$ (ℓ_d is a constant, since each 2^{-l_i} cube can be covered by at most $(\sqrt{d})^d$ cubes of diameter 2^{-l_i}). To make things non-intersecting, let us, as usual, make our cubes semi-open. WLOG, by scaling, can assume $K \subset Q_0$ - 1-dyadic cube.

Fix $n > 0$. Let $L_n := \{Q : Q - 2^{-n} \text{ cube} : Q \cap K \neq \emptyset\}$

Let us define μ_n on $\bigcup_{Q \in L^n} Q \supset K$ the following way.

μ_n' would have constant density on each $Q \in L_n$, with $\mu_n'(Q) = h(2^{-n})$.

Then $\forall Q \in L_{n-1}$. If $\mu_n'(Q) \leq h(2^{-n+1})$, then let $\mu_n^2 = \mu_n'$ in Q. If not, let, for $E \subset Q$, $\mu_n^2(E) = \frac{h(2^{-n+1})}{\mu_n'(Q)} \mu_n'(E)$ - rescale μ'.

keep doing the rescaling till we constructed $\mu_n := \mu_n^n$. μ_n is supported on $\overline{\bigcup_{Q \in L_n} Q}$.

Call a 2^{-k}-cube Q good if $\mu_n(Q) = h(2^{-k})$. Good cubes cover $\bigcup_{Q \in L_n} Q$ (since for any x we can look at the last time we rescaled μ when we replaced the measure). So we can select non-intersecting cover by them, N_n, and

$\mu_n(\bigcup_{Q \in L_n} Q) \geq \mu(\bigcup_{Q \in N_n} Q) = \sum_{Q \in N_n} h(2^{-k(Q)}) \geq \frac{H_h(K)}{\ell_d}$; and $\mu_n(\bigcup_{Q \in L_n} Q) \leq h(1)$.

By construction, $\forall$ any 2^{-m} cube Q with $m \leq n$, we have $\mu_n(Q) \leq h(2^{-m})$.

By Banach-Alaoglu Thm, $\exists$ subsequence n_i: such that $\mu_{n_i} \to \mu$ weak*, i.e. $\forall$ continuous φ, $\int \varphi \, d\mu_{n_i} \to \int \varphi \, d\mu$.

Then μ is supported on $K = \bigcap_n \left(\bigcup_{Q \in C_n} Q\right)$

$$\mu(K) \leq \mu(Q_0) = \lim_{n\to\infty} \int 1 \, d\mu_n = \lim_{n\to\infty} \mu_n(Q_0) \geq \frac{H_h(K)}{C_d}.$$

Fix now a 2^{-m} cube Q, and let φ be continuous, equal to 1 on Q, and $=0$ outside of 2^{-m} nbhd of Q. Then note that $\int \varphi \, d\mu_n \leq 3^d h(2^{-m})$ (since 2^{-m} nbhd of Q can be covered by 3^d 2^{-m}-cubes). So

$\mu(Q) \leq \int \varphi \, d\mu = \lim_{n\to\infty} \int \varphi \, d\mu_n \leq 3^d h(2^{-m})$.

Now any $B(x,r)$ can be covered by some C_d diadic cubes of size $< r$. Thus $\mu(B(x,r)) \leq 3^d \cdot C_d \, h(r)$.

By rescaling μ by C_d, we get the required measure ∎

Potential theory and dimension

June 19, 2017 9:15 AM

Let $K_\alpha(x) = \begin{cases} |x|^{-\alpha}, & \alpha>0 \\ -\log|x|, & \alpha=0 \end{cases}$. Let μ be a Borel measure in $\mathbb{R}^d$. The α-potential of μ is defined as

$$U_\alpha^\mu(x) := \int_{\mathbb{R}^d} K_\alpha(x-y)\,d\mu(y).$$

Special role is played by $\alpha = d-2$. In this case, $\Delta K_{d-2}(x) = \sigma_d \delta_0$, for some σ_d depending on d $\left(\sigma_d = \begin{cases} -2\pi, & d=2 \\ -(d-2)\,\mathrm{Area}(S^d) \end{cases}\right)$.

so $\Delta U_\alpha^\mu = \sigma_d \mu$ — solution to a Laplace problem.

The α-energy of μ:

$$I_\alpha(\mu) := \iint K_\alpha(x-y)\,d\mu(x)d\mu(y) = \int U_\alpha^\mu(x)\,d\mu(x).$$

Familiar electrostatic energy for $\alpha=1$, $d=3$.
The integral always exists, for compactly supported μ, since $K_\alpha(x-y)$ is bounded bellow on supp μ. Can be ∞.

Let E be a compact set. The α-equilibrium constant of E is defined as $V_\alpha := \inf\limits_{\mathrm{supp}\,\mu\subset E,\ \mu(E)=1} I_\alpha(\mu)$. $V_\alpha=\infty \iff \forall \mu: \mathrm{supp}\,\mu\subset E$, $I_\alpha(\mu)=\infty$.

α-capacity of E is defined as
$$\mathrm{Cap}_\alpha(E) = \begin{cases} e^{-V_\alpha}, & \alpha=0 \\ V_\alpha^{-1/\alpha}, & \alpha>0 \end{cases}$$
$\mathrm{Cap}_\alpha E>0 \iff \exists \mu: \mathrm{supp}\,\mu\subset E,\ I_\alpha(\mu)<\infty$

I will not talk about multitude of interesting properties of capacity. Instead, I will concentrate on connections with dimension.

Thm (Frostman) Let E - compact, $H_h(E)>0$ for some h with $\int_0 \frac{h(t)}{t^{1+\alpha}}\,dt<\infty$. Then $\mathrm{Cap}_\alpha(E)>0$.

In particular, if $\mathrm{Cap}_\alpha(E)=0$, then for any $\beta>\alpha$, $H_\beta(E)=0$, and thus $\mathrm{Hdim}\,E\le\alpha$.

Proof. By Frostman's lemma, $\exists$ h-smooth μ.
Let us show that $I_\alpha(\mu)<\infty$ by showing that $\|U_\alpha^\mu\|_\infty<\infty$.
Let $n(t) := \mu(B(x,t)) \le C h(t)$. $R := \mathrm{diam}\,E$.
Then (at least for $\alpha>0$):
$$V_\alpha(x) = \int_0^R t^{-\alpha}\,dn(t) = \lim_{\varepsilon\to 0}\left(\left(\frac{n(t)}{t^\alpha}\right)\Big|_\varepsilon^R + \alpha\int_\varepsilon^R \frac{n(t)\,dt}{t^{\alpha+1}}\right) \le$$
$$\frac{n(R)}{R^\alpha} + \alpha\int_0^R \frac{n(t)\,dt}{t^{\alpha+1}} \le \frac{h(R)}{R^\alpha} + C\alpha\int_0^\infty \frac{h(t)\,dt}{t^{\alpha+1}}$$

Same for $\alpha=0$.
For the other direction, we'll need a lemma:

Lemma on singularity Let E be compact in $\mathbb{R}^d$, $H_h(E)<\infty$ for some gauge h, and $E_1 := \{x\in E : \lim \frac{\mu(Q_n(x))}{h(b^{-n})} = 0\}$, where $Q_n(x)$ is b^{-n}-biadic cube containing x. Then $\mu(E_1)=0$.

Proof. Let $\varphi_n(x) := \frac{\mu(Q_n(x))}{h(b^{-n})}$. Then on E_1, $\varphi_n\to 0$.
By Egoroff's thm, if $\mu(E_1)>0$, $\exists$ $E_2\subset E_1$, $\mu(E_2)>0$ and $\varphi_n\to 0$ uniformly on E_2. Take n so that $\varphi_n(x)<\varepsilon$ $\forall x\in E_2$.
One can find an
cover of E by b-cubes with sides $\le b^{-n}$, and $\sum h(\mathrm{diam}\,Q_k)\le 2b^d H_h(E)$. Then $\mu(E_2)\le \sum\limits_{Q_k\cap E\ne\emptyset}\mu(Q_k) < \varepsilon\sum h(\mathrm{diam}\,Q_k) \le 2b^d\varepsilon H_h(E) \to 0$

Thm. Let E be compact, $m_\alpha(E)<\infty$ ($\alpha>0$), $m_{1/\log t}(E)<\infty$ ($\alpha=0$). Then $\mathrm{Cap}_\alpha(E)=0$.

Corollary. For $\alpha>\mathrm{Hdim}\,E$, $\mathrm{Cap}_\alpha E=0$. $\alpha<\mathrm{Hdim}\,E$ (by previous Thm). $\mathrm{Cap}_\alpha(E)>0$. $\mathrm{Hdim}\,E = \inf\{\alpha : \mathrm{Cap}_\alpha E=0\} = \sup\{\alpha : \mathrm{Cap}_\alpha(E)>0\}$.

Proof Let μ - probability measure on E ...

$\mathrm{Cap}_\alpha(E)>0$. $\quad H\dim E = \inf\{\alpha : \mathrm{Cap}_\alpha E = 0\} =$ ''
$\sup\{\alpha : \mathrm{Cap}_\alpha(E)>0\}$.

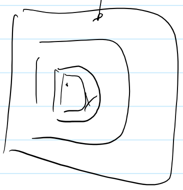
Proof. Let μ – probability measure on E.

Let $h_\alpha(t) = \begin{cases} t^\alpha, & \alpha>0 \\ \frac{1}{\log\frac{1}{t}}, & \alpha=0\end{cases}$

E_1 – singularity set from previous lemma with $h=k_\alpha$.
$x \notin E_1 \iff \exists\, Q_{n_i}(x), \eta(x)>0 : \mu(Q_{n_i}(x)) > \eta(x) h_\alpha(6^{-n_i})$.
Let us prove that (x) $V^\mu_\alpha(x) = \infty$. Then, since $\mu(E_1)=0$,
$I_\alpha(\mu) = \int V^\mu_\alpha(x)\, d\mu(x) = \infty$.
If $\mu(x)>0$, it is obvious. If not, select $\varepsilon_i > 0$
for $Q_i := \{y \in Q_{n_i}(x) : |y-x|>\varepsilon_i\}$, $\mu(Q_i) > \frac{1}{2}\eta(x) k_\alpha(6^{-n_i})\}$

Large max.

Taking subsequence, we can assume that $Q_j \cap Q_k = \emptyset$, $j\neq k$.



So $V^\mu_\alpha(x) \geq \sum_i \int_{Q_i} k_\alpha(x-y)\,d\mu(y)$

If $\alpha>0$, $\int_{Q_i} k_\alpha(x-y)\,d\mu(y) \geq (\sqrt{d}\,6^{-n_i})^{-\alpha}\mu(Q_i) \geq d^{-\alpha/2} 6^{n_i\alpha}(6^{-n_i})^{\alpha}\cdot\frac{1}{2}\eta(x) \geq \frac{d^{-\alpha/2}}{2}\eta(x)$. So $\sum_i = \infty$.
Similarly, for $\alpha=0$.

Let us see the precision of this for Cantor sets with side sequence ℓ_n in $\mathbb{R}^d$.

Thm. Let E be (ℓ_n) – Cantor set in $\mathbb{R}^d$. Then
$\mathrm{Cap}_\alpha E > 0 \iff \sum 2^{-nd} k_\alpha(\ell_n) < \infty$.

Metric>0 $\Downarrow$ Cap>0

Pf. Let $\sum 2^{-nd} k_\alpha(\ell_n) < \infty$. Let us prove $(\Leftarrow)$ for $\alpha=0$. Define $h(t) = \begin{cases} 2^{-n}, & t=\ell_n \\ \text{linear } \ell_n<t<\ell_{n-1}\end{cases}$. Then

$$\int_0 \frac{h(t)}{t}\,dt \leq \sum \int_{\ell_{n-1}}^{\ell_n} \frac{2^{-n+1}}{t}\,dt = \sum 2^{-n+1}\log\frac{\ell_n}{\ell_{n-1}} \leq 2\sum 2^{-n}\log\frac{1}{\ell_n} < \infty.$$

and the standard dyadic measure is h-smooth, so $H_h(E)>0$ for some h with $\int_0 \frac{h(t)}{t^{1+\alpha}}\,dt < \infty \implies \mathrm{Cap}_\alpha(E)>0$.

On the other hand, for any measure μ, $\operatorname{supp}\mu = E$, and $I^\alpha(\mu)<\infty$, we have:

$$I^\alpha(\mu) = \int d\mu(y)\int k_\alpha(x-y)\,d\mu(x) = \int d\mu(y)\Big(\int_{r=0} k_\alpha(r)\,d\mu(B(y,r))\Big)$$

$$\overset{\text{parts}}{\asymp} \int d\mu(y)\Big(\int \mu(B(y,r))\frac{dr}{r^{1+\alpha}}\Big) \geq$$

$$\sum_{n=0} \int_{\ell_n}^{\ell_{n-1}} \frac{dr}{r^{1+\alpha}} \underbrace{\int \mu(B(y,\ell_n))\,d\mu(y)}_{\text{there are essentially } 2^{nd} \text{ disks like that}} \geq \sum_{n=0}^{\infty}\Big(k_\alpha(\ell_n)\cdot\sum_{k=1}^{2^{nd}}\mu(Q_k^n)^2\Big) \geq$$

By Cauchy, $\sum_{k=1}^{2^{nd}} \mu(Q_k^n)^2 \geq 2^{-nd}\underbrace{(\mu(E))^2}_{=2^{-nd}}$ $\Big(\frac{\sum a_i^2}{n} \geq \big(\frac{\sum a_i}{n}\big)^2\Big)$.

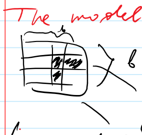
Thus $\infty > I^\alpha(\mu) \gtrsim \sum_{n=0}^{\infty} k_\alpha(\ell_n)2^{-nd}$. Thus
$\mathrm{Cap}_\alpha(E)>0 \implies \sum k_\alpha(\ell_n)2^{-nd} < \infty$. ∎

Dimension of a random Cantor sets

June 8, 2017 4:35 PM

Let us apply the potential theory to prove the following result about random Cantor sets. $b \in \mathbb{N}$, $b \geq 2$.

The model:



b^d cubes in $\mathbb{R}^d$ in Q_0-unit cube. keep each with probability p, discard otherwise. Repeat in each cube we kept to

to get C_2. & $C = \cap C_n$.

Thm $p \leq b^{-d} \Rightarrow C = \emptyset$ a.s.

$p > b^{-d} \Rightarrow C$ is a.s. either empty or

$H\dim C = M\dim C = d + \log_b p$. The latter occurs with positive probability.

Upper bound

Lemma (Easy bound on Minkowski) Let K be a random set and $\overline{\lim} \frac{\log \mathbb{E}(N(K,\varepsilon))}{|\log \varepsilon|} < \alpha$, $\alpha > 0$. Then a.s. $\overline{M\dim}\, K \leq \alpha$

Pf. Take $\alpha_1 > \alpha_2 > \alpha$. Then, for small ε,

$E(N(K, 2^{-n})) \leq 2^{-n\alpha_2}$. So $\mathbb{P}(N(K,2^{-n}) \geq 2^{n\alpha_1}) \leq 2^{-n\alpha_1} E(N(K,2^{-n}) \mid N(K,2^{-n}) \geq 2^{n\alpha_1}) \leq$

$2^{-n(\alpha_1 - \alpha_2)}$ So, by Borel-Cantelli, a.s. $\overline{\lim} \frac{N(K,2^{-n})}{n \log 2} \leq \alpha_1$ ∎

Return to random Cantor sets.

For any kept cube, let

$q_k = \binom{b^d}{k} p^k (1-p)^{b^d - k}$ be the probability that we kept exactly k cubes inside it. Expected number of the first level kept subcubes is

$m := p b^d = \sum_{k=0}^{b^d} k q_k$.

Let $Z_n(Q)$ is the number (random) of subcubes of b-adic Q of size $b^{-\ell}$ kept after n steps. Then, by induction on n,

$\mathbb{E}(Z_n(Q) \mid Q \text{ is kept}) = m^{n-\ell}$

In particular, $E(N(C, \sqrt{d}\, b^{-n})) \leq m^n$ so by Lemma,

$\overline{M\dim}\, C \leq \max(\log_b m, 0) = \max(d + \log_b p, 0)$ a.s.

Lower bound.

One can proceed by using the obvious measure on C (define inductively on C_n, pass to the limit) and use Mass Distribution Principle. But we are talking about something happening locally w.r.t. random measure. Too complicated. Instead, let us use the energy of this measure.

Lemma (b-adic friendly formula for capacity). ($\alpha > 0$)

For $K \subset Q_0$, $I_\alpha(\mu) \asymp \sum_{n=0}^{\infty} \left[b^{n\alpha} \left(\sum_{Q - b^{-n} \text{ cubes}} \mu(Q)^2 \right) \right]$

Pf. On one hand

$I_\alpha(\mu) \geq \sum_{n=0}^{\infty} k_\alpha(b^{-n}) (\mu \times \mu) \{(x,y) : b^{-n-1} < |x-y| \leq b^{-n}\} =$ by parts

$\sum_{n=0}^{\infty} \left(k_\alpha(b^{-n}) - k_\alpha(b^{-n+1})\right) \mu\times\mu \{|x-y| \leq b^{-n}\} \gtrsim$

$\sum_{n=0}^{\infty} b^{n\alpha} \sum_{Q - b^{-(n-\ell)}\text{-cube}} \mu(Q)^2$, where $b^\ell \geq \sqrt{d}$.

On the other hand,

$I_\alpha(\mu) \leq \sum_{n=0}^{\infty} k_\alpha(b^{-n}) \mu\times\mu \{(x,y) : b^{-n} \leq |x-y| < b^{-n+1}\} \lesssim$ sum by parts

$\sum b^{n\alpha} \mu\times\mu \{(x,y) : |x-y| \leq b^{1-n}\}$.

We say that b^{-n} cube Q_1 is adjacent to Q_2 if they are neighbors. Notation: $Q_1 \sim Q_2$.

Then

$\mu\times\mu \{(x,y) : |x-y| \leq b^{1-n}\} \leq \sum_{\substack{Q_1 \sim Q_2 \\ Q_1, Q_2 - b^{1-n} \text{ cubes}}} \mu(Q_1)\mu(Q_2)$.

Now, observe first that

$\mu(Q_1)\mu(Q_2) \leq \frac{\mu(Q_1)^2 + \mu(Q_2)^2}{2}$, and second, that

$\#\{Q_1 : Q_1 \sim Q\} = 3^d$.

So, we have

$\mu\times\mu \{(x,y) : b^{-n} < |x-y| \leq b^{1-n}\} \leq 3^d \sum_{Q - b^{1-n}\text{ cube}} \mu(Q)^2$ ∎

Let us return to the random set C.

Let $Z_n(Q)$ be the number of b^{-n} cubes in C which are descendants of Q. $Z_n := Z_n(Q_0)$.

Then $E Z_n = m^n$, and for Q being a b^{-k}-cube, ($k < n$)

$Z_n(Q)$ has the same distribution as Z_{n-k}.

Define now a (random) measure μ on C by $\mu(Q) = \lim m^{-n} Z_n(Q)$.

descendants of Q. $Z_n := Z_n(Q_0)$.
Then $E Z_n = m^n$, and for Q being a b^{-k}-cube, $(k < n)$
$Z_n(Q)$ has the same distribution as Z_{n-k}.
Define now a (random) measure μ on C by $\mu(Q) = \lim_{n\to\infty} m^{-n} Z_n(Q)$.
It is indeed a measure, since $\sum_{\substack{Q \cap C \\ Q \subset Q_1 \\ Q - b^{-k'}\text{-cube}}} \mu(Q) = \mu(Q_1)$ for any b^{-k}-cube Q, (since it was true for any $n > k$ and $m^{-n} Z_n(Q)$))
But why does the limit exist?

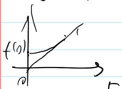
Lemma. $m^{-n} Z_n \to \mu(Q_0)$ a.s. and in L^2 for $m > 1$.
The sets $\{\omega : \mu(Q_0) = 0\}$ and $\{\omega : C = \emptyset\}$
are the same up to a set of probability 0.
When $m \le 1$, these sets are of full probability
When $m > 1$, these sets are of $p < 1$.

Assume lemma, then
$I_\alpha(\mu) \asymp \sum_{n=0}^{\infty} b^{-n(\alpha+1)} \sum_{Q - b^{-n}\text{-cube}} \mu(Q)^2$. Observe that
$\mu(Q)$ has the same distribution as $m^{-n}\mu(Q_0)$, so
$E\, I_\alpha(\mu) = \sum_{n=0}^{\infty} b^{dn} \cdot m^{-2n} E(\mu(Q_0)^2) P(Q \in C_n) =$
The last probability is exactly p^n (need to make the pick of Q and its ancestors n times), so
$= E(\mu(Q_0)^2) \sum_{n=0}^{\infty} b^{dn} (p b^d)^{-2n} p^n = E(\mu(Q_0)^2) \sum_{n=0}^{\infty} b^{n(\alpha - d - \log_b p)}$.
Converges for $\alpha < d + \log_b p$! So $I_\alpha(\mu)$ is a.s. finite.
Thus, if $\mu(C) > 0$, μ gives a measure on C with $I_\alpha(\mu) < \infty$. By lemma, it is the same, a.s. as $\{C \neq \emptyset\}$.

Pf of Lemma.

independent guys $E\left(\left(\sum a_i\right)^2\right) = \sum E(a_i^2)$, if $E(a_i) = 0$.

Let $K = E((Z_1 - m)^2) < \infty$ and
Then $E(|Z_{n+1} - m Z_n|^2) = \sum_{k=0}^{b^{nd}} E(|Z_{n+1} - m Z_n|^2 \mid Z_n = j) P(Z_n = j) \overset{2}{=}$
$\sum_{k=0}^{b^n} (j\, E((Z_1 - m)^2)) \cdot P(Z_n = j) = K \sum j\, P(Z_n = j) = K E(Z_n) = K m^n$.
Then $E\left(\left(m^{-n} Z_n - m^{-n-1} Z_{n+1}\right)^2\right) = m^{-2n-2} E(|Z_{n+1} - m Z_n|^2) = K m^{-2} m^{-n}$.
So $\sum E\left(\left(m^{-n} Z_n - m^{-n-1} Z_{n+1}\right)^2\right) < \infty$, which means $\exists \lim_{n\to\infty} m^{-n} Z_n$ in L^2.
The same inequality imply a.s. convergence,
since
$E \sum |m^{-n} Z_n - m^{-n-1} Z_{n+1}| = \sum E \le \sum \left(E\left(\left(m^{-n} Z_n - m^{-n-1} Z_{n+1}\right)^2\right)\right)^{1/2} < \infty$.
so $\sum$ converges a.s.
Now, notice that: $\{\omega : C = \emptyset\}$ is the same as $\{\exists n : Z_n = 0\}$.
Since obviously $\{\omega : \mu(C) = 0\} \supset \{\omega : C = \emptyset\}$, we just need to show that the sets have the same probability.
For this, consider
generating polynomial: $f(x) := \sum_{k=0}^{b^d} q_k x^k$
Note: $f(1) = 1$, $f(0) = q_0 = 1 - p$, $f'(1) = \sum k q_k = m$.
Observe also that $f''(x) = \sum k(k-1) q_k x^{k-2} \ge 0$, so f is convex.
Also $f(x) = E(x^{Z_1})$.
But $Z_2 = \sum_{i=1}^{Z_1} \xi_i$, with ξ_i distributed like Z_1 and independent, so
$E(x^{Z_2}) = E(x^{\xi_1 + \dots + \xi_{Z_1}}) = \sum E(x^{\xi_1 + \dots + \xi_k}) P(Z_1 = k) =$
$\sum E(x^{\xi_1}) \dots E(x^{\xi_k}) P(Z_1 = k) = \sum q_k f(x)^k = f(f(x))$.
and, in general, by induction, the generating polynomial of Z_n is
$E(x^{Z_n}) = f^{\circ n}(x)$ - n-th iteration of f!
So $P(Z_n = 0) = f^{\circ n}(0)$,



$f^{\circ n}(0)$ converges to the smallest fixed point of $f(x)$ at $[0,1]$. If $m \le 1$, there is only one such fixed point, 1, so $P(C = \emptyset) = 1$.
For $m > 1$, $\exists$ another fixed point, x_0. So
$P(C = \emptyset) = x_0$.
Now, if $r := P(\mu(C) = 0)$, then
$r = \sum_k P(\mu(C) = 0 \mid Z_1 = k) P(Z_1 = k) = \sum r^k q_k = f(r)$.
We know that $E(\mu(C)) = \lim_{n\to\infty} E(m^{-n} Z_n) = 1$, so $r < 1$,
thus $r = x_0$. ∎

The Billingsley Lemma. Dimension of measures. Dimension of an ergodic measure for a holomorphic dynamical system.

June 8, 2017 4:33 PM

Now, let us generalize Mass Distribution Principle.
For $x \in \mathbb{R}^d$, let $Q_n(x)$ be b^{-n} b-adic cube containing x, as before.

Lemma (Billinsley)
Let K be a Borel set in $\mathbb{R}^d$, μ-finite Borel measure on $\mathbb{R}^d$ and $\mu(K)>0$. If for some $\beta \geq \alpha \geq 0$
$\alpha \leq \lim_{n\to\infty} \frac{\log \mu(Q_n(x))}{-n\log b} \leq \beta$ $\forall x \in K$, then $\alpha \leq \mathrm{Hdim} K \leq \beta$.

Pf. Take $\alpha_1 < \alpha$. By the left inequality,
$\lim_{n\to\infty} b^{\alpha_1 n} \mu(Q_n(x)) = 0$ $\forall x \in K$. By Singularity lemma applied to $h(x)=x^{\alpha_1}$, $m_h(K)=\infty$, so $\mathrm{Hdim} K \geq \alpha_1$ $\forall \alpha_1<\alpha$.
In the other direction, for $\beta_1>\beta$, we have
$\overline{\lim}\, b^{\beta_1 n}\mu(Q_n(x)) = \infty$ for all $x \in K$.
Fix $\varepsilon>0$, and for any $x \in K$ let $n(x)$ be the smallest n such that $b^{\beta_1 n}\mu(Q_n(x)) > (\sqrt{d})^{\beta_1}$ and $\sqrt{d}\, b^{-n} \leq \varepsilon$.
$(Q_{n(x)})_{x\in K}$ is a cover of K. Select a non-intersecting subcover (Q_k). Then $\mathrm{diam}\, Q_k < \varepsilon$ (by the choice of $n(x)$),
$\sum (\mathrm{diam}\, Q_k)^{\beta_1} \leq \sum \mu(Q_k) \leq \mu(\mathbb{R}^d)$. Thus
$m_{\beta_1,\varepsilon}(K) \leq \mu(\mathbb{R}^d)$. Take $\varepsilon\to 0$, to get that $m_{\beta_1}(K)<\infty$ ∎

An application: dimension of lacunary set.
Let $S \subset \mathbb{N}$ - an infinite set.
Define $A_S = \{x = \sum_{k\in S} x_k 2^{-k},\ x_k \in \{0,1\}\}$.
Can be thought of as a Cantor set: start with $C_0=[0,1]$, for each $k \in S$, exclude intervals with 1 at k-th digit to get C_k from C_{k-1}, for $k \notin S$, $C_k = C_{k-1}$.
A_S is covered by $2^{\#(S\cap\{1,2,\dots,n\})}$ dyadic intervals of size 2^{-n}. So
$\overline{\mathrm{Mdim}}\, A_S = \overline{\lim} \frac{\log N(A_S, 2^{-n})}{\log 2^n} = \overline{\lim} \frac{\#(S\cap\{1,\dots,n\})}{n} =: \overline{d}(S)$ - the upper density of S
$\underline{\mathrm{Mdim}}\, A_S = \underline{\lim} \frac{\#(S\cap\{1,\dots,n\})}{n} =: \underline{d}(S)$ - the lower density of S.
Let us show that
$\mathrm{Hdim}\, A_S = \underline{d}(S)$.
Let us define μ to be the measure giving equal weight $2^{-\#(S\cap\{1,\dots,n\})}$ to each of the n-th generation intervals of C_n. For $x \in C$,
$\underline{\lim}_{n\to\infty} \frac{\log \mu(Q_n(x))}{\log 2^{-n}} = \underline{d}(S)$. So, by Billingsley, $\mathrm{Hdim}\, A_S = \underline{d}(S)$ ∎

Billingsley's lemma leads to the notion of

Def Dimension of a measure: μ - Borel measure in $\mathbb{R}^d$.
$\dim \mu = \inf\{\dim A : \mu(A^c)=0,\ A \subset \mathbb{R}^n$ - Borel$\}$
Another, equivalent, def
$\dim \mu = \inf\{\alpha : \mu \perp m_\alpha\}$. Remark. $\inf = \min$, take A_n with $\mu(A_n^c)=0$, $\mathrm{Hdim}\, A_n < \dim\mu + \frac{1}{n}$). $A := \cap A_n$.

Local dimension of μ at x is defined as
$\dim_\mu^b(x) = \underline{\lim}_{n\to\infty} \frac{\log \mu(Q_n(x))}{-n\log b}$.

Remark This definition, strictly speaking, depends on b.
An independent definition: $\dim_\mu(x) = \underline{\lim}_{r\to 0} \frac{\log \mu(B(x,r))}{|\log r|}$. Since $\mu(B(x,r)) \geq \mu(Q_n(x))$ for some n with $b^{-n} \asymp r$, we always have $\dim_\mu(x) \leq \dim_\mu^b(x)$.
An example of μ be uniform on $[0,\frac{1}{2}]$, shows that at $\frac{1}{2}$, $\dim_\mu^2 \frac{1}{2} = \infty$, $\dim_\mu = 1$.
Equivalent b-adic definition: consider $\max_{x\in Q} \mu(Q)$.
Also easy to see, using a version of Billingsley Lemma, that $\mu\{x : \dim_\mu^b(x) \neq \dim_\mu(x)$ for some $b\} = 0$.
We'll work with $\dim_\mu^b$, to simplify the derivations.

Lemma $\dim \mu = \operatorname{ess\,sup} \dim_\mu^b(x)$ ($\operatorname{ess\,sup} f := \inf\{M : \mu\{f(x) \geq M\} = 0\}$).
Pf. Pick $\alpha > \operatorname{ess\,sup} \dim_\mu(x)$, let
$A := \{x : \underline{\lim}_{n\to\infty} \frac{\log \mu(Q_n(x))}{-n\log b} \leq \alpha\}$. $\mu(A^c)=0$, and, by Billingsley, $\mathrm{Hdim}\, A \leq \alpha$.
If $\alpha < \operatorname{ess\,sup}$, $B := \{x : \underline{\lim}_{n\to\infty} \frac{\log \mu(Q_n(x))}{-n\log b} \geq \alpha\}$.
$\mu(B)>0$. If $\mu(E^c)=0$, then
$\mu(B) = \mu(E\cap B) > 0$ and $\underline{\lim} \geq \alpha$ on $E \cap B$.
By Billingsley, $\mathrm{Hdim}\, E \geq \mathrm{Hdim}\, E\cap B \geq \alpha$ ∎

An application, and a first example of multifractal analysis: Eggleston's thm.
Let us remind that $T_2(x) = 2x \bmod 1 : [0,1] \to [0,1]$.
If $x = \sum x_k 2^{-k}$, then $x_k = \chi_{[\frac{1}{2},1]}(2^k x \bmod 1) = \chi_{[\frac{1}{2},1]}(T_2^{\circ k}(x))$.
T_2 preserves Lebesgue measure, $|T_2^{-1}(B)| = |B|$, and
i) ergodic, i.e. if B is T_2 invariant $(T_2^{-1}(B) = B)$, then $|B| = 1$ or 0.

Then, by the famous Birkhoff ergodic theorem, for any $f \in L^1$, Then, by the famous Birkhoff ergodic theorem, for any $f \in L^1$,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} f(T^{\circ k}x) \to \int_0^1 f(t)\,dt \quad \mu\text{-a.s.}$$

In particular, for $f = \chi_{[\frac{1}{2},1]}$, $\frac{\sum_{k=1}^{n} x_k}{n} \to \frac{1}{2}$ a.s. by μ.

Let us look at an exceptional set

$A_p := \{x : \frac{\sum x_k}{n} \to p\}$, $p \neq \frac{1}{2}$, $0 \le p \le 1$, and try to compute its dimension.

To this end, define measure μ_p. It is enough to define it for the binary intervals:

$\mu_p([j2^{-n}, (j+1)2^{-n})) = p^{k(j)}(1-p)^{n-k(j)}$, where $k(j)$ is the number of 1_s in the binary expansion of j.

Alternatively, define inductively by assigning probability p to the right interval, $1-p$ – to the left: I.e. $\mu_p(Q_n(x)) = p^{\sum_{k=1}^{n} x_k}(1-p)^{\sum_{k=1}^{n}(1-x_k)}$.

$\underset{1-p\ \ p}{\vdash\!\!\!+\!\!\!\dashv}$ $p = \frac{1}{2}$ – Lebesgue measure.

Define $h_2(p) := -p\log_2 p - (1-p)\log_2(1-p)$ the entropy of μ_p with respect to T_2. I will return to the meaning of this later, but for now let us prove that

Lemma $H\dim A_p = \dim \mu_p = h_2(p)$.

Proof. First, let us observe that μ_p is also T_2-invariant and ergodic.

Thus μ_p-a.s. $\frac{1}{n}\sum x_k \to \int_0^1 \chi_{[\frac{1}{2},1]}(t)\,d\mu_p(t) = p$

Thus $\mu_p(A_p) = 1$.

$\dim \mu_p = \operatorname{ess\,sup} \dim_\mu(x)$. But

$$\frac{\log \mu_p(Q_n(x))}{-n\log 2} = \frac{1}{\log 2}\left(\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right)\log\frac{1}{p} + \left(\frac{1}{n}\sum_{k=1}^{n}(1-x_k)\right)\log\frac{1}{1-p}\right).$$

So, a.s. $n\to\infty$, since μ-a.e. $x \in A_p$, it converges to

$$\frac{p\log\frac{1}{p} + (1-p)\log\frac{1}{1-p}}{\log 2} = h_2(p).$$

So $h_2(p) = \dim \mu_p \le \dim A_p$. But for any $x \in A_p$, $\dim_{\mu_p}(x) = h_2(p)$, by above. So, by Billingsley, $\dim A_p \le h_2$, since $\mu_p(A_p) > 0$. ∎ Observe that $\dim_{\mu_p} x$ exists μ_p-a.e. as a limit, not $\underline{\lim}$.

Remark. A generalization:

$\vec{p} = (p_0, \dots, p_{b-1})$ – b-multiindex, with $p_0 + \dots + p_{b-1} = 1$, $p_j \ge 0$. Then define

$\mu_{\vec{p}}(Q_n(x)) = \prod_{j=1}^{n} p_{x_j}$, $x = \sum x_j b^{-j}$ – b-ary expansion.

$A_{\vec{p}} := \{x : \frac{\#\{k \le n,\ x_k = j\}}{n} \to p_j,\ j = 0, \dots, b-1\}$

Then, exactly as above,

$\dim \mu_{\vec{p}} = H\dim A_{\vec{p}} = h_b(\vec{p}) = -\sum_{j=0}^{b-1} p_j \log p_j$

What is the dynamic meaning of the entropy? Consider again $b=2$.

For x, consider $\frac{\mu(T_2(B(x,r)))}{\mu(B(x,r))}$ for small r, or, equivalently, $\frac{\mu_p(T_2(Q_n(x)))}{\mu_p(Q_n(x))}$. But $T_2(Q_n(x)) = Q_{n-1}(T_2 x)$, so the measure is divided by p (if $x_1 = 1$) or $(1-p)$ (if $x_1 = 0$).

Let us define Jacobian of μ $J_\mu := \lim_{n\to\infty} \frac{\mu_p(T_2(Q_n(x)))}{\mu_p(Q_n(x))} = \begin{cases} \frac{1}{p}, & \text{if } x_1 = 1 \\ \frac{1}{1-p}, & \text{if } x_1 = 0 \end{cases}$

One can compute $\mu_p(Q_n(x)) \approx J_\mu^{-1}(x) \cdot J_\mu^{-1}(T_2 x) \dots J_\mu^{-1}(T_2^{\circ n-1}x)\,\mu_p[0,1]$, so

$\frac{\log \mu_p(Q_n(x))}{n} = -\frac{1}{n}\sum \log(J_{\mu_p}(T_2^{\circ k}x)) \to -\int \log J_{\mu_p}\,d\mu_p$ – ergodic theorem!

$= -p\log\frac{1}{p} + (1-p)\log\frac{1}{1-p}$ – average expansion of μ. Notation: entropy of μ, $h_\mu(T) = h_\mu$. On average, $\log \mu(T_2 B(x,r)) - \log \mu(B(x,r)) = h_\mu(T)$.

On the other hand, average log expansion of distance is $\log 2$ – the map is linear, so it is easy. But let us rewrite it as $\lambda(\mu,T) = \int \log|T_2'|\,d\mu_p = \log 2$ – the Lyapunov exponent of μ.

By Ergodic Theorem, $\lambda(\mu) = \frac{1}{n}\sum_{k=0}^{n-1} \log T_2'(T_2^{\circ k}(x))$ for μ_p-a.e. x.

So, heuristically, after one iteration $\log\mu$ increases by h_μ, and the radius increases by $\lambda(\mu)$.

So, again heuristically, we get the right result here: $\dim_\mu(x) = \frac{h_\mu}{\lambda_\mu}$ μ-a.e., and, consequently, $\dim \mu = \frac{h_\mu}{\lambda_\mu}$.

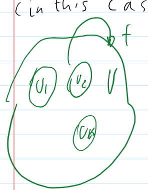
Turns out, that it has far-reaching generalizations. Let me state a 1D version.

Thm (Mañé–Przytycki) (Volume lemma).

Let $X \subset \mathbb{C}$ – compact, $f: X \to X$ – analytic in some nbhd of X, μ – f-invariant ergodic measure on X, $\lambda(\mu) > 0$ (so, on μ-average, f is exponentially expanding).

Then μ-a.e. $\dim_\mu(x) = \frac{h_\mu}{\lambda_\mu} = \lim_{r\to 0}\frac{\log \mu(B(x,r))}{|\log r|}$ (exists as a limit!)

Let us prove it for a special case: conformal Cantor sets (in this case, it is a result of Manning).



Def. Let $U_1, \dots U_k \Subset U$ – open ($A \Subset B \Leftrightarrow \bar{A} \subset B$)

$f: \cup U_j \to U$ – conformal, onto. $T_j := f|_{U_j}^{-1}$

$C = \bigcap (f^{\circ j})^{-1}(U)$ – conformal Cantor set.

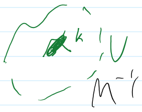
Can represent pts $x \in C$ as sequences $(i_1, \dots, i_n, \dots)$, $i_j \in \{1, \dots, k\}$ – because each pt. appears in $\bigcap_n T_{i_1} \circ \dots \circ T_{i_n}(U) = C_{i_1 \dots i_n}$

We will need the Egg-Yolk principle:

Let $K \subset U$, K-compact, U-open.

because each pt. appears ... $!\, T_{i_1}\circ\dots\circ T_{i_n}(U) =: C_{i_1\dots i_n}$

We will need the **Egg-Yolk principle:**



Let $K \subset U$, K-compact, U-open.
Then $\exists M(K,U)$, such that for <u>any</u> conformal $\varphi : U \to \mathbb{C}$ and $x \in K$,

$$M^{-1}(K,U) \le \frac{\operatorname{diam} \varphi(K)}{|\varphi'(x)|} \le M(K,U).$$

Because of this, $\operatorname{diam}(C_{i_1\dots i_n}) \asymp |(T_{i_1}\circ\dots\circ T_{i_n})'(x)| \asymp |f'(\underbrace{T_{i_1}\dots T_{i_n}(x)}_{f(T_{i_1}\dots T_{i_n}(x))}) \cdot f'(T_{i_2}\dots T_{i_n}(x))\dots|^{-1}$

for any $x \in C$.

so

$$\lim_{n\to\infty} \frac{\log \operatorname{diam} C_{i_1\dots i_n}(y)}{n} = -\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \log |f'(f^{\circ k}(y))| = \int \log|f'(x)|\, d\mu(x) \text{ if } \mu \text{ is } f\text{-ergodic.}$$

$$\mu(C_{i_1\dots i_n}(y)) \asymp \left(\frac{\mu(C)}{\mu(C_{i_n})} \cdot \frac{\mu(C_{i_n})}{\mu(C_{i_n i_{n-1}})} \cdot \dots\right)^{-1} \asymp \left(J_\mu(y)\, J_\mu(f(y)) \dots J_\mu(f^{\circ(n-1)}(y))\right)^{-1}$$ – integrals here, actually, but the result is the same.

so

$$\lim_{n\to\infty} \frac{\log \mu(C_{i_1\dots i_n}(y))}{n} = -\frac{1}{n}\sum \log J_\mu(f^{\circ k}(y)) \to h_\mu.$$

so μ-a.e. $\dim_\mu(y) = \dfrac{h_\mu}{\lambda_\mu}$.

What about $\operatorname{Hdim} C$?

Pick $x \in C$, and let, as before, $x_{i_1\dots i_n} = T_{i_1}\circ\dots\circ T_{i_n} x$.
Then for a fixed $t \ge 0$,

$$\sum_{i_1,\dots,i_n} (\operatorname{diam} C_{i_1\dots i_n})^t \asymp \sum |(f^{\circ n})'(x_{i_1\dots i_n})|^{-t}$$

Define $P(t) := \lim_{n\to\infty} \dfrac{\log \sum |(f^{\circ n})'(x_{i_1\dots i_n})|^{-t}}{n}$. This limit exists, does not depend on x (since it measures the same metric thing). It is called **Pressure spectrum.**

$P(0) = \log k$, $P(\infty) = -\infty$, $P(t)$ is strictly decreasing, thus $\exists!\, t_0 : P(t_0) = 0$.

Pick $t > t_0$. Then for any $\varepsilon > 0\ \exists n : \operatorname{diam} C_{i_1\dots i_n} \le \varepsilon$ for all such sets. Thus C is covered by $C_{i_1\dots i_n}$, and, since $P(t) < 0$, $\lim_{n\to\infty} \sum (\operatorname{diam} C_{i_1\dots i_n})^t = 0$. so $m_t(C) = 0$, thus $\operatorname{Hdim} C \le t_0$.

It is correct upper bound, and one can show it by constructing special measures μ_t which are invariant, ergodic, and $\mu_t(C_{i_1\dots i_n}) \asymp |(f^{\circ n})'(x_{i_1\dots i_n})|^{-t} \cdot e^{-P(t)n} = (\operatorname{diam} C_{i_1\dots i_n})^t e^{P(t)n}$ – **Gibbs measures.**

In particular, for $t = t_0$, $\mu_{t_0}(C_{i_1\dots i_n}) \asymp (\operatorname{diam} C_{i_1\dots i_n})^{t_0}$, so μ_{t_0} is t_0-smooth, and we can use Billingsley's lemma.

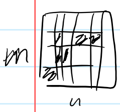
Hint for construction:
give $C_{i_1\dots i_n}$ mass $|(f^{\circ n})'(x_{i_1\dots i_n})|^{-t}$. Does not quite work – need to normalise by $e^{-P(t)n}$ to make it sum up to 1 (up to subexponential correction).

The actual construction is much more sophisticated!

Dimension of self-affine sets

June 8, 2017 4:37 PM

Now I will give an example of set where
$\mathrm{Hdim}\, K < \mathrm{Mdim}\, K$ "non-trivially".
Let D be an $n\times m$ pattern on the square:



Formally, $D \subset \{0,\dots,n-1\}\times\{0,\dots,m-1\}$
Repeat the pattern on every non-removed rectangle.

$K(D) := \{ \sum_{k=1}^{\infty} (a_k n^{-k}, b_k m^{-k}) : (a_k, b_k) \in D \}$.

Most famous: McMullen set, corresponding to $D = \{(0,0),(1,1),(2,0)\}$ in 3×2 rectangle.

Lemma (Easy). Assume $n > m$ and that every row contains a chosen rectangle (i.e. $\forall j\ \exists i : (i,j)\in D$).
Then $\mathrm{Mdim}\, K(D) = 1 + \log_n \frac{\#(D)}{m}$.

Exercise: What happens when not every row contains a chosen rectangle?

Proof. Let $r := \#(D)$ and let us look at the j-th stage of the construction. It consists of r^j elongated rectangles of width n^{-j} and height m^{-j}.

Let $k = \left[\frac{\log n}{\log m} j\right]$. Then we need
$\frac{m^{-j}}{n^{-j}} = m^{k-j}$ squares of size m^{-k} to cover each such rectangle, and we cannot use less than m^{k-j-1} (because every row has a rectangle in it, thus the height of $K(D)\cap$ any given rectangle is still m^{-j}).
So $N(K(D), m^{-k}) \approx r^j m^{k-j}$,

$$\mathrm{Mdim}\, K(D) = \lim_{k\to\infty} \frac{\log r^j + \log m^{k-j}}{k \log m} =$$

$$\frac{\log r}{\log n} + \frac{\log n}{\log n} - \frac{\log m}{\log n} = 1 + \log_n \frac{r}{m}$$

So, for McMullen set,
$\mathrm{Hdim}\, K(D) = 1 + \log_3 \frac{3}{2}$.

But

Thm. In the assumptions of the lemma,
$\mathrm{Hdim}(K(D)) = \log_m \left(\sum r(j)^{\log_n m} \right)$ where
$r(j) = \#\{ i : (i,j) \in D\}$ - number of rectangles lying in a given row.

It is easy to see that
$\mathrm{Hdim}(K(D)) \le \mathrm{Mdim}(K(D))$, with equality reached iff $r(j)$ does not depend on j.

For McMullen set, $\dim K(D) = \log_2 (1 + 2^{\log_3 2})$

Pf Instead of b-adic squares, we use more appropriate approximate squares

$\alpha := \frac{\log m}{\log n}$

$$Q_k(x,y) = \left\{ (x',y') : \begin{array}{ll} x' = \sum x'_j n^{-j} & x'_j = x_j,\ j \le \lfloor \alpha k \rfloor \\ y' = \sum y'_j m^{-j} & y'_j = y_j,\ j \le k \end{array} \right\}.$$

We expand x in base n! y in base m!

It is almost a square, since $m^{-k} \le n^{-\lfloor \alpha k\rfloor} \le n m^{-k}$,
and $\mathrm{diam}\, Q_k \cong m^{-k}$ (up to a factor of $\sqrt{2}\, n$).
Given a probability vector $\vec{p}$ with coordinates

$p(d)$, $d\in D$, define the probability measure $\mu_{\vec p}$ on $K(D)$ by the usual iterative process. Assume also that $p(d_1)=p(d_2)$ if d_1, d_2 are in the same row. Then

$$\mu_{\vec p}\, Q_k(x,y) = \prod_{j=1}^{k} p(x_j, y_j) \prod_{j=\lfloor \alpha k\rfloor+1}^{k} r(y_j)$$

(since $Q_k(x,y)$ contains exactly $\prod_{j=\lfloor\alpha k\rfloor+1}^{k} r(y_j)$ rectangles of level k, each of the same measure).

Thus $\log(\mu_{\vec p}(Q_k(x,y))) = \sum_{j=1}^{k} \log p(x_j, y_j) + \sum_{j=\lfloor\alpha k\rfloor+1}^{k} \log r(y_j)$

Then, again by ergodic Thm,

$$\lim_{k\to\infty} \frac{1}{k} \log(\mu_{\vec p}(Q_k(x,y))) = \sum_{d\in D} p(d)\log p(d) + (1-\alpha)\sum_{d\in D} p(d)\log r(d)$$

and, by a version of Billingsley's Lemma for approximate squares,

$$\dim(\mu_{\vec p}) = \frac{1}{\log m} \sum p(d)\left(\log\frac{1}{p(d)} + \log\left(r(d)^{\alpha-1}\right)\right).$$

To maximize it, pick

$$p(d) = \frac{1}{Z} r(d)^{\alpha-1}, \quad Z = \sum_{d\in D} r(d)^{\alpha-1} = \sum_{j=0}^{m-1} r(j)^{\alpha}.$$

Then $\dim\mu = \log_m Z$, and, thus we obtain the right lower bound for $H\dim K(D)$.

For the upper bound, if

$$S_k(x,y) := \sum_{j=1}^{k} \log r(y_j).$$

$\frac{1}{k} S_k(x,y)$ is uniformly bounded,

$$\log\mu(Q_k(x,y)) = \sum_{j=1}^{k} \log\frac{1}{Z} r(y_j)^{\alpha-1} + \left(\sum_{j=1}^{k} \log r(y_j) - \sum_{j=1}^{\lfloor\alpha k\rfloor} \log r(y_j)\right) =$$

$$= -\sum_{j=1}^{k} \log Z + (\alpha-1) S_k(x,y) + S_k(x,y) - S_{\lfloor\alpha k\rfloor}(x,y), \text{ or}$$

$$\log\mu(Q_k(x,y)) + k\log Z = \alpha S_k(x,y) - S_{\lfloor\alpha k\rfloor}(x,y).$$

so: $$\frac{1}{\alpha k}\log\mu(Q_k(x,y)) + \frac{1}{\alpha}\log Z = \frac{S_k(x,y)}{k} - \frac{S_{\lfloor\alpha k\rfloor}(x,y)}{\alpha k}.$$

Now do the telescoping series to get that, since

$$\sum \frac{S_{\alpha^{-j}}(x,y)}{\alpha^{-j}} - \frac{S_{\alpha^{-j+1}}(x,y)}{\alpha^{-j+1}} \text{ converges,}$$

$$\overline{\lim}\left(\frac{S_k(x,y)}{k} - \frac{S_{\lfloor\alpha k\rfloor}(x,y)}{\alpha k}\right) \geq 0. \Rightarrow$$

$$\overline{\lim}\left(\log\mu(Q_k(x,y)) + k\log Z\right) \geq 0 \Rightarrow$$

$$\underline{\lim}\ \frac{\log\mu(Q_k(x,y))}{-k} \leq \log Z. \text{ so, since}$$

$\operatorname{diam} Q_k(x,y) \asymp m^{-k}$, $H\dim(K(D)) \leq \log_m Z$

(we use again an approximate square version of Billingsley's Lemma)